Comment on Identical Motion in Classical and Quantum Mechanics

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Abstract

Makowski and Konkel [Phys. Rev. A **58**, 4975 (1998)] have obtained certain classes of potentials which lead to identical classical and quantum Hamilton-Jacobi equations. We obtain the most general form of these potential.

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In their recent paper [1], Makowski and Konkel study the class of potentials allowing for a quantum potential Q which only depends on time,

$$Q = K(t)\ . \tag{1}$$

Since the quantum effects are related to the quantum force $-\nabla Q$, the corresponding systems have identical classical and quantum dynamics. In the polar representation where the wave function takes the form $\Psi(\vec{x}, t) = R(\vec{x}, t) \exp[i\varphi(\vec{x}, t)/\hbar]$ the Schrödinger equation is written as

$$\frac{\partial R^2}{\partial t} + \nabla \cdot \left(R^2 \nabla \frac{\varphi}{m}\right) = 0\ , \tag{2}$$

$$\frac{\partial \varphi}{\partial t} + \frac{(\nabla\varphi)^2}{2m} + V + Q = 0\ . \tag{3}$$

Here R and φ are real-valued functions and the quantum potential Q is defined by $Q(\vec{x}, t) := -\hbar^2 \nabla^2 R/(2mR)$.

The classification of all the potentials with the above mentioned property is equivalent to finding the general solution of Eqs. (1), (2) and (3) for the three real unknowns, R, φ, and V. The complete classification for the case $K = 0$ is given in Ref. [2] which predates the work of Makowski and Konkel [1]. In Ref. [2] the analogous problem for Klein-Gordon equation is also addressed. Moreover a new semiclassical perturbation theory around these potentials is outlined, and its application to quantum cosmology is discussed.

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The results of Makowski and Konkel [1] can also be generalized to give the complete classification of potentials which allow for identical classical and quantum dynamics in arbitrary dimensions. Following Ref. [2], we shall refer to these potential as *semiclassical potentials*.

As explained by Makowski and Konkel [1], for the case of stationary states, where $\partial R/\partial t = 0$ and $\varphi(\vec{x}, t) = -Et + S(\vec{x})$, Eq. (2) reduces to

$$\nabla \cdot (R^2 \nabla S) = 0 \ . \tag{4}$$

Makowski and Konkel [1] obtain a class of solutions of this equation by solving

$$R^2 \nabla S = \text{const} \ . \tag{5}$$

Therefore, they restrict their analysis to a set of particular solutions of Eq. (4). This is, however, not necessary. The general solution of Eq. (4) can be easily obtained by making the following change of dependent variable:

$$S \to \tilde{S} := RS \ . \tag{6}$$

Let us first introduce

$$\lambda := \frac{\sqrt{2mK}}{\hbar} \ ,$$

in terms of which Eq. (1) takes the form

$$\nabla^2 R + \lambda^2 R = 0 \ . \tag{7}$$

Now substituting $S = \tilde{S}/R$ in (4) and making use of Eq. (7), we obtain

$$\nabla^2 \tilde{S} + \lambda^2 \tilde{S} = 0 \ . \tag{8}$$

Therefore in view of Eqs. (3), (6), and (7), the most general potential allowing for identical classical and quantum dynamics for a stationary state is given by

$$V = E - \frac{1}{2m} \left\{ (\hbar\lambda)^2 + \left[\nabla \left(\frac{\tilde{S}}{R} \right) \right]^2 \right\} \ , \tag{9}$$

where R and $\tilde{S}$ are solutions of Eqs. (7) and (8), respectively. Note that Eq. (9) is valid in any number of dimensions.

More generally, we can use the analog of the change of variable (6), namely

$$\varphi \to \tilde{\varphi} := R\varphi \ , \tag{10}$$

to handle the general problem, where the wave function Ψ does not represent a stationary state. In this case, Eq. (7) still holds, but λ is a function of time. Using (10) and (7), Eqs. (2) and (3) take the form

$$\nabla^2 \tilde{\varphi} + \lambda^2 \tilde{\varphi} = -2m \frac{\partial R}{\partial t} \ , \tag{11}$$

$$V = -\frac{\partial}{\partial t} \left(\frac{\tilde{\varphi}}{R} \right) - \frac{1}{2m} \left\{ (\hbar\lambda)^2 + \left[\nabla \left(\frac{\tilde{\varphi}}{R} \right) \right]^2 \right\} \ . \tag{12}$$

Therefore the set of all the semiclassical potentials are classified by the solutions of Eqs. (7) and (11). Note that these equations are not evolution equations. Eq. (7) may be viewed as a constraint equation in which t enters as a parameter through the dependence of λ on t. Once the boundary conditions of this equation are chosen, it can be solved using the known methods of solving linear partial differential equations with 'constant' coefficients. The solution is then used to evaluate the right hand side of Eq. (11). The latter is a nonhomogeneous linear partial differential equation. It can be solved using the well-known Green's function methods. Again it is the boundary conditions that determine the solution.

The above analysis shows that it is the choice of the function K or alternatively λ together with the boundary conditions of Eqs. (7) and (11) that determine the set of potentials which allow for identical classical and quantum dynamics.

We wish to conclude this article by the following remarks.

1.) Makowski and Konkel conclude their paper [1] emphasizing that "*A number of additional potentials would be found if new solutions of Eq. (2a)* [This is our Eq. (2)] *were obtained. This, however, can be a difficult task.*" We have shown that a simple change of the dependent variable φ, namely $\varphi \to \tilde{\varphi} := R\varphi$, eases this 'difficulty' and leads to a complete classification of all such potentials.

2) Following the above remark, Makowski and Konkel write: "*Among the potentials derived here we have not found any example from the known set of potentials implying bound states, e.g., Coulomb, Morse, or Pöschl-teller. This likely follows from the fact that most stationary states of physical interest have no classical limit.*" In this connection, we must emphasize that for the semiclassical potentials leading to identical classical and quantum dynamics, the amplitude $R := |\Psi|$ of the wave function of a stationary state satisfies Eq. (7). This is just the eigenvalue equation for the Laplacian in $\mathbb{R}^n$. It is well-known that this equation does not admit a solution corresponding to a bound state.

3) It is well-known that shifting the Hamiltonian H by a time-dependent multiple of the identity operator, i.e.,

$$H \to H' = H + f(t) \ . \tag{13}$$

leaves all the physical quantities of the system invariant [3]. This is in fact true both in quantum and classical mechanics. In quantum mechanics, such a transformation corresponds to a phase transformation

$$\Psi \to \Psi' = e^{i\zeta(t)}\Psi \tag{14}$$

of the Hilbert space, where $\zeta(t) := -\int_0^t f(t')dt'$. Therefore, it leaves the quantum states, the expectation values of the observables, and the excitation energies invariant. It also leaves the quantum potential invariant. But it does change the classical potential according to

$$V \to V' = V + f(t)\ . \tag{15}$$

In fact, the effect of such a transformation on Eqs. (2) and (3) is the shift (15) of the potential.

Now consider the case that the quantum potential is a function of time, $Q = K(t)$. Then we can make a phase transformation (14) of the Hilbert space with $\zeta(t) = \int_0^t K(t')dt'$ in (14), so that $f = -K$ and the total potential $V + Q = V + K$ in Eq. (3) is transformed to $V' + Q - K = V$. Therefore, the dynamics of a state with time-dependent quantum potential and a state with a zero quantum potential are equivalent. In general, we can make a phase transformation of the Hilbert space which effectively removes such a quantum potential. Therefore, as far as the physical quantities are concerned the case $Q = K(t)$ is equivalent to the case $Q = 0$. The latter has been thoroughly studied in Ref. [2].

References

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